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BONDED HALF PLANES CONTAINING AN ARBITRARILY ORIENTED CRACK†

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Abstract—The plane elastostatic problem for two bonded half planes containing an arbitrarily oriented crack in the neighborhood of the interface is considered. Using Mellin Transforms the problem is formulated as a system of singular integral equations. The equations are solved for various crack orientations, material combinations, and external loads. The numerical results given in the paper include the stress intensity factors, the strain energy release rates, and the probable cleavage angles giving the direction of crack propagation.

1. INTRODUCTION

The structural stength of composite materials is controlled to a considerable extent by the size, shape, orientation, and distribution of the flaws and imperfections which exist in the material. Usually these flaws and imperfections exhibit themselves in the form of entrapped gas or weak impurities on the interface, ruptured bonds, cracks, inclusions, and geometric singularities arising from the particular shape of the constituent materials. From the view-point of fracture initiation and propagation in the medium particularly important are the manufacturing flaws such as flat cavities which develop during bonding or casting, small cracks resulting from the residual stresses, and fatigue cracks caused by the cyclic nature of the external loads. Thus, in studies relating to the fracture initiation and propagation in the material, it is necessary to have a good estimate of those factors representing the severity of the external loads in the neighborhood of the "isolated dominant flaw". Generally, the comparison of these factors (such as the stress intensity factors, the strain energy release rate, the crack opening stretch, or the cleavage stress at a characteristic distance from the flaw boundary) with the corresponding characteristic constant representing the resistance of the material to fracture constitutes the fracture criterion. If the dominant imperfection is completely imbedded in a homogeneous phase and is located sufficiently far from the phase boundaries or interfaces, then the disturbance of the stress field around the imperfection will not be affected by the neighboring phases and the disturbed stress field may be obtained by solving the problem for an infinite homogeneous solid. On the other hand if the flaw is located near a phase boundary or a bi-material interface, then the solution of the problem for the nonhomogeneous medium becomes necessary.

An up-to-date review of the available solutions for variety of crack and inclusion geometries in composite materials may be found in [1]. The primary interest of this paper is in the evaluation of the disturbed stress field around a crack located near a bi-material interface or, as a limiting special case, free boundary. The problem was studied in a previous series of

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papers for two special crack orientations, namely the case of a crack located parallel to or at the interface [2–6] and the problem of a crack perpendicular to and crossing the interface [7, 8]. In this paper we will assume that the orientation of the crack with respect to the interface (i.e. the angle θ_0 and the distance d in Fig. 1) is arbitrary and the problem is one of plane strain or generalized plane stress. As in the previous studies it will also be assumed

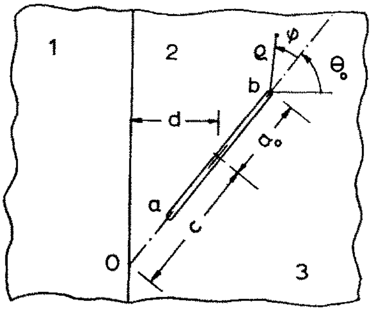

Fig. 1. Notation for the inclined crack.

that the interface in the non-homogeneous medium is either a plane or has a sufficiently large radius of curvature so that the disturbed stress field can be approximated by that of a crack in two bonded elastic half planes (Fig. 1). Even though the problem will be formulated and solved for the bonded half planes, the technique described in the paper appears to be quite general and may be used to treat the problem of any number of bonded wedges with radial cracks.

2. THE INTEGRAL EQUATIONS OF THE PROBLEM

Using the conventional superposition technique the solution of the problem of a traction-free crack in the composite medium under a given set of external loads can be expressed as the sum of two solutions: the first obtained for the given external loads and the given medium without the crack, and the second obtained for the two bonded half planes with a crack where the only external loads are the crack surface tractions which are equal and opposite to the stresses found in the first solution on the presumed location of the crack. It is clear that only the second solution which gives the disturbed stress field due to the existence of the crack will have singularities. Also note that in the second problem, since the external loads are local and statically self-equilibrating, in the application of Mellin transforms the regularity conditions required of the solution as $r \to \infty$ will be satisfied. Following now, for example [9], in polar coordinates the plane elasticity problem for a medium having the elastic constants μ and κ ($\kappa = 3$–4ν for plane strain, $\kappa = (3 - \nu)/(1 + \nu)$ for plane stress) may be formulated as

$$\sigma(r, \theta) = \tau_{r\theta} + i\tau_{\theta\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \chi}{\partial \theta}\right) + i\frac{\partial^2 \chi}{\partial r^2},$$

$$\tau_{rr} = \frac{1}{r^2}\frac{\partial^2 \chi}{\partial \theta^2} + \frac{1}{r}\frac{\partial \chi}{\partial r},$$

$$v(r,\theta)=\frac{\partial u_r}{\partial r}+i\frac{\partial u_\theta}{\partial r}=\frac{1}{2\mu}\left\{-\frac{\partial^2\chi}{\partial r^2}+\frac{i}{r^2}\frac{\partial\chi}{\partial\theta}-\frac{i}{r}\frac{\partial^2\chi}{\partial r\,\partial\theta}+\frac{1+\kappa}{4}\left[\frac{\partial\psi}{\partial\theta}+2ir\frac{\partial\psi}{\partial r}+r\frac{\partial^2\psi}{\partial r\,\partial\theta}+ir^2\frac{\partial^2\psi}{\partial r^2}\right]\right\},\qquad \text{(1a–c)}$$

$$\nabla^4\chi=0,\qquad \nabla^2\psi=0,\qquad \frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right)=\nabla^2\chi.\qquad \text{(2a–c)}$$

Referring to Fig. 1, the medium will be considered as consisting of three infinite wedges: 1 (μ_1,κ_1), $\frac{\pi}{2}<\theta<\frac{3\pi}{2}$; 2 (μ_2,κ_2), $\theta_0<\theta<\frac{\pi}{2}$; 3 (μ_2,κ_2), $-\frac{\pi}{2}<\theta<\theta_0$. For an infinite wedge with an arbitrary angle, using the Mellin transforms to solve (2), from (1) we obtain

$$\begin{aligned}\mathscr{M}[r^2\sigma,s]&=\textstyle\sum(s,\theta)\\&=2i(s+1)[Ase^{is\theta}+B(s+1)e^{i(s+2)\theta}-\bar{B}e^{-i(s+2)\theta}],\\ \mathscr{M}[r^2\tau_{rr},s]&=-s(s+1)(Ae^{is\theta}+\bar{A}e^{-is\theta})\\&\quad-(s+1)(s+4)[Be^{i(s+2)\theta}+\bar{B}e^{-i(s+2)\theta}],\\ \mathscr{M}[r^2v,s]&=V(s,\theta)\\&=\frac{s+1}{\mu}[Ase^{is\theta}+B(s+1)e^{i(s+2)\theta}+\kappa\bar{B}e^{-i(s+2)\theta}],\end{aligned}\qquad \text{(3a–c)}$$

where A and B are functions of the transform variable s. The Mellin transform of a function $f(r)$ defined and suitably regular in $(0<r<\infty)$, and its inverse are defined by

$$F(s)=\mathscr{M}[f,s]=\int_0^\infty f(r)r^{s-1}\,\mathrm{d}r,$$

$$f(r)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}F(s)r^{-s}\,\mathrm{d}s,\qquad \text{(4a,b)}$$

where c is such that $r^{c-1}f(r)$ is absolutely integrable in $(0,\infty)$. The transform of derivatives may be shown to be

$$\int_0^\infty r^n\frac{\mathrm{d}^nf(r)}{\mathrm{d}r^n}r^{s-1}\,\mathrm{d}r=(-1)^n\frac{\Gamma(s+n)}{\Gamma(s)}F(s),\qquad (5)$$

provided

$$r^{s+m-1}\frac{\mathrm{d}^{m-1}f}{\mathrm{d}r^{m-1}}\to 0\quad\text{as}\quad r\to(0,\infty),\qquad (m=1,\dots,n).\qquad (6)$$

In applications conditions (6) provide the information to determine the strip of regularity containing the line $Re(s)=c$ in the inversion integral.†

† It should be noted that in working with Mellin transforms up to the inversion stage in the manipulations the transform variable s is treated as a real variable. The function $F(s)$ is analytically extended into the complex plane from the real line when the inversion is evaluated. In the present problem, the complex notation in (1 and 3) is used only for convenience. Thus, for example, in separating the transforms of the stress components $\tau_{r\theta}$ and $\tau_{\theta\theta}$ in (3a), s should be treated as real and A and B should be treated as complex quantities.

If we now let the subscripts 1, 2, 3 stand for the wedges shown in Fig. 1. the problem must be solved under the following boundary conditions:

$$\sigma_1(r, \pi/2) = \sigma_2(r, \pi/2), \quad (0 \le r < \infty),$$
$$v_1(r, \pi/2) = v_2(r, \pi/2), \quad (0 \le r < \infty), \tag{5a,b}$$
$$\sigma_1(r, 3\pi/2) = \sigma_3(r, -\pi/2), \quad (0 \le r < \infty),$$
$$v_1(r, 3\pi/2) = v_3(r, -\pi/2), \quad (0 \le r < \infty), \tag{6a,b}$$
$$\sigma_2(r, \theta_0) = \sigma_3(r, \theta_0), \quad (0 \le r < \infty), \tag{7}$$
$$\lim_{\theta \to \theta_0 + 0} \sigma_2(r, \theta) = p_2(r) + ip_1(r), \quad (a < r < b), \tag{8}$$
$$v_2(r, \theta_0) = v_3(r, \theta_0), \quad (0 \le r < a, b < r < \infty), \tag{9}$$
$$\int_a^b [v_2(r, \theta_0) - v_3(r, \theta_0)]\, dr = 0. \tag{10}$$

where σ_j and v_j, $(j = 1, 2, 3)$ are defined by (1), $\mu_2 = \mu_3$, and note that (9) and (10) correspond to the condition of displacement continuity outside the crack. In the solution of the problem as given by (3) there are six unknown (complex) functions $A_j(s)$, $B_j(s)$, $(j = 1, 2, 3)$ to be determined. The homogeneous conditions (5–7) provide five equations. The sixth equation is obtained from the mixed conditions given by (8) and (9). Thus, eliminating five of the unknown functions, the problem may be formulated as a system of dual integral equations for two unknown real functions by using (8) and (9). However, a somewhat more direct method to solve the problem would be its reduction to a system of singular integral equations for a pair of real functions f_1 and f_2 defined by†

$$v_2(r, \theta_0 + 0) - v_3(r, \theta_0 - 0) = f(r) = f_2(r) + if_1(r), \quad (a < r < b). \tag{11}$$

If (8) and (9) are replaced by

$$v_2(r, \theta_0) - v_3(r, \theta_0) = \begin{cases} f(r), & (a < r < b), \\ 0, & (0 \le r < a,\ b < r < \infty), \end{cases} \tag{12}$$

and if we define

$$U(s) = \mathscr{M}[r^2\{v_2(r, \theta_0) - v_3(r, \theta_0)\}, s] = \int_a^b f(r) r^{s+1}\, dr$$
$$= \int_a^b [f_2(r) + if_1(r)] r^{s+1}\, dr = U_2(s) + iU_1(s), \tag{13}$$

by substituting from (3) into (5–7) and (13) we obtain six linear algebraic equations in A_j and B_j, $(j = 1, 2, 3)$, which may be solved giving $A_j(s)$ and $B_j(s)$ in terms of $U(s)$. For example, for the wedge 2 we find

$$A_2(s) = \frac{\mu_2}{(1 + \kappa_2)s(s + 1)(e^{i\pi s} - e^{-i\pi s})}\{[m_1(s + 1)se^{is\theta_0}$$
$$+ m_1(s + 1)^2 e^{i(s+2)\theta_0} - se^{-is(\pi+\theta_0)} + m_2 e^{i(s+2)\theta_0}]U_2(s)$$
$$+ [m_1(s + 1)(s + 2)e^{is\theta_0} + m_1(s + 1)^2 e^{i(s+2)\theta_0}$$
$$+ (s + 2)e^{-is(\pi+\theta_0)} + m_2 e^{i(s+2)\theta_0}]iU_1(s)\},$$

† Note that physically f_1 and f_2 represent the densities of edge dislocations distributed along $\theta = \theta_0$ in two bonded half planes.

$$B_2(s) = \frac{\mu_2}{(1+\kappa_2)(s+1)(e^{-i\pi s} - e^{i\pi s})}\{[-m_1 s e^{is\theta_0} - m_1(s+1)e^{i(s+2)\theta_0} - e^{-i(s\theta_0+2\theta_0+s\pi)}]U_2(s) - [m_1(s+2)e^{is\theta_0} + m_1(s+1)e^{i(s+2)\theta_0} - e^{-i(s\theta_0+2\theta_0+s\pi)}]iU_1(s)\}, \quad (14a,b)$$

where

$$m_1 = (m-1)/(\kappa_2 m + 1), \qquad m = \mu_1/\mu_2,$$
$$m_2 = (\kappa_1 - m\kappa_2)/(\kappa_1 + m). \quad (15)$$

Expressions similar to (14) may be found for wedges 1 and 3.

All the field quantities in wedge 2 may now be obtained in terms of $f(r)$ by substituting from (14) into (3) and using the inversion formula. In particular, from (14), (3a), (8), and (4b) we obtain

$$\frac{1+\kappa_2}{\mu_2}[p_2(r) + ip_1(r)] = \lim_{\theta\to\theta_0+0}\frac{1}{\pi}\int_a^b [K_2(r, r_0, \theta)f_2(r_0) + iK_1(r, r_0, \theta)f_1(r_0)]\,dr_0, \qquad (a < r < b), \quad (16)$$

$$K_2(r, r_0, \theta) = \int_{c-i\infty}^{c+i\infty}\frac{ds}{e^{i\pi s} - e^{-i\pi s}}\frac{r_0^{s+1}}{r^{s+2}}\{m_1 s(s+1)e^{is(\theta+\theta_0)} + [m_2 + m_1(s+1)^2]e^{i(s\theta+s\theta_0+2\theta_0)} - se^{is(\theta-\theta_0-\pi)} + m_1 s(s+1)e^{i(s\theta+s\theta_0+2\theta)} + m_1(s+1)^2 e^{i(s+2)(\theta+\theta_0)} + (s+1)e^{i(s\theta+2\theta-s\theta_0-2\theta_0-s\pi)} + m_1 s e^{-i(s\theta+2\theta+s\theta_0)} + m_1(s+1)e^{-i(s+2)(\theta+\theta_0)} + e^{i(s\theta_0+2\theta_0-s\theta-2\theta+s\pi)}\},$$

$$K_1(r, r_0, \theta) = \int_{c-i\infty}^{c+i\infty}\frac{ds}{e^{i\pi s} - e^{-i\pi s}}\frac{r_0^{s+1}}{r^{s+2}}\{m_1(s+1)(s+2)e^{is(\theta+\theta_0)} + [m_2 + m_1(s+1)^2]e^{i(s\theta_0+s\theta+2\theta_0)} + (s+2)e^{is(\theta-\theta_0-\pi)} + m_1(s+1)(s+2)e^{i(s\theta+s\theta_0+2\theta)} + m_1(s+1)^2 e^{i(s+2)(\theta+\theta_0)} - (s+1)e^{i(s\theta+2\theta-s\theta_0-2\theta_0-s\pi)} - m_1(s+2)e^{-i(s\theta+s\theta_0+2\theta)} - m_1(s+1)e^{-i(s+2)(\theta+\theta_0)} + e^{i(s\theta_0+2\theta_0-s\theta-2\theta+s\pi)}\}. \quad (17a,b)$$

In (16) the order of integrations has been changed. For $\theta > \theta_0$, since the related integrals are uniformly convergent, this is permissible. (16) provides a system of integral equations to determine the unknown functions f_1 and f_2. To solve this system the kernels K_1 and K_2 must be evaluated which may be done either by using the residue theory and expressing them as infinite series or by reducing the integrals to real integrals and evaluating them numerically. In either case, it is first necessary to determine the strip of regularity containing the constant c. Let the integrands in (17) be analytically extended into the entire plane and let the poles s_j be ordered as

$$\cdots < Re(s_{-2}) < Re(s_{-1}) < Re(s_{+1}) < Re(s_{+2}) < \cdots. \quad (18)$$

Let $Re(s_{-1}) < c < Re(s_{+1})$. Thus, the integrals in (17) may be evaluated by closing the contour to the left for $r < r_0$ and to the right for $r > r_0$ by means of semicircles of infinite radius, and by summing the residues. Noting that for $r \to 0$, $\tau_{ij} \sim 0(r^{-\lambda})$, $Re(\lambda) < 1$, and for $r \to \infty$, $\tau_{ij} \sim 0(r^{-\alpha})$, $Re(\alpha) \geq 1$, and since the residues are of the form $r^{-(s_j+2)}$, s_j then should be ordered such that

$$Re(s_{-1}) < -1, \qquad Re(s_{+1}) \geq -1. \tag{19}$$

Thus, $Re(s_{-1}) < Re(s) = c < -1$ gives the strip of regularity, s_{-1} being the first pole to the left of the line $Re(s) = -1$.

For an arbitrary value of θ_0, even though the residues in (17) may be evaluated without any difficulty, the resulting infinite series cannot be summed in closed form. Hence, it is difficult to study the singular behavior of the kernels. For this reason in this paper the kernels will be evaluated by reducing (17) to real integrals. For this we let $c = -1$, $s = -1 + iy$, $(-\infty < y < \infty)$, and indent the contour in such a way that the pole $s_{+1} = -1$ lies to the right of the line of integration. The integrals in (17) may then be expressed as

$$K_k(r, r_0, \theta) = \int_{c-i\infty}^{c+i\infty} H_k(s)\,\mathrm{d}s = \int_{-\infty}^{\infty} H_k(-1+iy)i\,\mathrm{d}y - \lim_{s\to -1} \pi i(s+1)H_k(s), \qquad (k = 1, 2). \tag{20}$$

Evaluating the residue at $s = -1$, from (17) it may be shown that

$$\lim_{s\to -1} \pi i(s+1)H_k(s) = \frac{1}{r}(m_2 - m_1 - 2), \qquad (k = 1, 2). \tag{21}$$

On the other hand, from (10) and (11) we have

$$\int_a^b f_k(r_0)\,\mathrm{d}r_0 = 0, \qquad (k = 1, 2). \tag{22}$$

Thus, when substituted into (16), the integrated terms in (20) will have no contribution. In the remaining integral in (20) the integrand H_k turns out to be the sum of an odd function and an even function† in y, giving K_j in terms of real integrals in $(0, \infty)$. As $r \to r_0$ these integrals become divergent. Since the integrands are bounded at $y = 0$, the divergent parts can be separated by considering the asymptotic behavior of the integrands as $y \to \infty$. By defining

$$\rho = \log(r_0/r), \qquad \varepsilon = \theta - \theta_0, \tag{23}$$

for small values of ε from (17) and (20) it may be shown that

$$\begin{aligned} K_k(r, r_0, \varepsilon) &= \frac{2}{r}\int_0^\infty \{e^{-\varepsilon y}[1 + 0(e^{-y(\pi-2\theta_0)})]\sin \rho y + 0(e^{-y(\pi-2\theta_0+\varepsilon)})\cos \rho y\}\,\mathrm{d}y \\ &= \frac{2}{r}\frac{\rho}{\rho^2+\varepsilon^2} + M_k(r, r_0, \varepsilon), \qquad \left(0 \leq \theta_0 < \frac{\pi}{2},\ k = 1, 2\right), \end{aligned} \tag{24}$$

where $M_k(r, r_0, 0)$ is bounded for all values of r and r_0 in the closed interval $[a, b]$.

Substituting now from (24) into (16), separating the real and imaginary parts, and letting $\varepsilon \to 0$ we obtain

† If not, it can always be put in that form.

$$\frac{1+\kappa_2}{2\mu_2}\,p_i(r)=\frac{1}{\pi}\int_a^b\sum_{j=1}^{2}\left[\frac{\delta_{ij}}{r\log(r_0/r)}+k_{ij}(r,r_0)\right]f_j(r_0)\,\mathrm{d}r_0,\qquad (i=1,2;\quad a<r<b),\tag{25}$$

where the bounded kernels k_{ij}, $(i,j=1,2)$ are given by

$$k_{11}(r,r_0)=\frac{1}{2r}\int_0^\infty\frac{\mathrm{d}y}{\sinh\pi y}\sin\rho y[\cosh 2\theta_0 y(m_1+4m_1y^2\cos^2\theta_0-m_2)$$
$$+2m_1y\sin 2\theta_0\sinh 2\theta_0 y+2e^{-\pi y}],$$

$$k_{12}(r,r_0)=\frac{1}{2r}\int_0^\infty\frac{\mathrm{d}y}{\sinh\pi y}[\cos\rho y\sinh 2\theta_0 y(4m_1y^2\cos^2\theta_0-m_1-m_2)$$
$$-4m_1y\cos^2\theta_0\sin\rho y\sinh 2\theta_0 y],$$

$$k_{21}(r,r_0)=\frac{1}{2r}\int_0^\infty\frac{\mathrm{d}y}{\sinh\pi y}[\cos\rho y\sinh 2\theta_0 y(m_1-4m_1y^2\cos^2\theta_0+m_2)$$
$$-4m_1y\cos^2\theta_0\sin\rho y\sinh 2\theta_0 y],$$

$$k_{22}(r,r_0)=\frac{1}{2r}\int_0^\infty\frac{\mathrm{d}y}{\sinh\pi y}\sin\rho y[\cosh 2\theta_0 y[(m_1+4m_1y^2\cos^2\theta_0-m_2)$$
$$-2m_1y\sin 2\theta_0\sinh 2\theta_0 y+2e^{-\pi y}],\qquad (a<(r,r_0)<b).\tag{26a–d}$$

In the system of integral equations (25) the dominant kernels have a simple Cauchy-type singularity. This may be seen by observing that

$$\frac{1}{r\log(r_0/r)}=\frac{1}{r\log\left[1+\left(\frac{r_0}{r}-1\right)\right]}$$
$$=\frac{1}{r\left(\frac{r_0}{r}-1\right)}\left[1+\sum_1^\infty\frac{(-1)^n}{n+1}\left(\frac{r_0}{r}-1\right)^n\right]^{-1}$$
$$=\frac{1}{r_0-r}\left[1+0\left(\frac{r_0}{r}-1\right)\right].\tag{27}$$

Thus, the system of singular integral equations is of the following conventional form:

$$\frac{1+\kappa_2}{2\mu_2}\,p_i(r)=\frac{1}{\pi}\int_a^b\sum_{j=1}^{2}\left[\frac{\delta_{ij}}{r_0-r}+h_{ij}(r,r_0)\right]f_j(r_0)\,\mathrm{d}r_0,\qquad (i=1,2;\quad a<r<b),$$

$$h_{ij}(r,r_0)=k_{ij}(r,r_0)+\left[\frac{1}{r\log(r_0/r)}-\frac{1}{r_0-r}\right]\delta_{ij},\qquad (i,j=1,2).\tag{28a,b}$$

The index of the integral equations (28) is $\kappa=1$; hence the solution will contain two arbitrary (real) constants which are determined from the additional conditions (22).

3. SOLUTION OF THE INTEGRAL EQUATIONS AND STRESS INTENSITY FACTORS

Referring to [10], it may be shown that the index of the system of singular integral equations is $+1$ and its solution is of the following form:

$$f_i(r)=g_i(r)[(b-r)(r-a)]^{-1/2},\qquad (a<r<b,\quad i=1,2),\tag{29}$$

where the unknown functions $g_i(r)$, $(i = 1, 2)$ are bounded in the closed interval $[a, b]$. Even though, in principle, the system can be regularized and reduced to a pair of Fredholm-type integral equations, its solution may be obtained with much less computational effort by using the technique described in [11]. In the numerical solution the main problem is the evaluation of the kernels k_{ij} given by (26) for which in this paper a modified version of Filon's integration formula has been used [12].

In the application of the results to fracture problems in composites, of particular interest are the stress intensity factors and the probable plane of cleavage at a given crack tip. The normal and shear components, k_1 and k_2, of the stress intensity factor are defined by and may be evaluated from the following expressions [2, 4, 5]:

$$k_1(a) = \lim_{r\to a} \sqrt{2(a-r)}\,\tau_{2\theta\theta}(r, \theta_0) = \lim_{r\to a} \frac{2\mu_2}{1+\kappa_2}\sqrt{2(r-a)}f_1(r),$$

$$k_1(b) = \lim_{r\to a} \sqrt{2(r-b)}\,\tau_{2\theta\theta}(r, \theta_0) = -\lim_{r\to b} \frac{2\mu_2}{1+\kappa_2}\sqrt{2(b-r)}f_1(r),$$

$$k_2(a) = \lim_{r\to a} \sqrt{2(a-r)}\,\tau_{2r\theta}(r, \theta_0) = \lim_{r\to a} \frac{2\mu_2}{1+\kappa_2}\sqrt{2(r-a)}f_2(r),$$

$$k_2(b) = \lim_{r\to b} \sqrt{2(r-b)}\,\tau_{2r\theta}(r, \theta_0) = -\lim_{r\to b} \frac{2\mu_2}{1+\kappa_2}\sqrt{2(b-r)}f_2(r). \qquad \text{(30a–d)}$$

The constants k_1 and k_2 are a measure of the intensity of the stresses around the crack tips. For example, at the crack tip $(r = b, \theta = \theta_0)$ the cleavage stress may be expressed as [13]

$$\sigma_{\phi\phi}(\rho, \phi) = \frac{1}{\sqrt{2\rho}} \cos\frac{\phi}{2}\left(k_1 \cos^2\frac{\phi}{2} - \frac{3}{2}k_2 \sin\phi\right) + O(\sqrt{\rho}), \qquad (31)$$

where (ρ, ϕ) are the polar coordinates at the crack tip, ϕ being measured from the line which is the prolongation of the crack (Fig. 1). Thus, once k_1 and k_2 are determined, for brittle solids the probable angle ϕ_c of crack propagation may be postulated as the angle of the radial plane corresponding to the maximum cleavage stress and may be determined from $\partial\sigma_{\phi\phi}/\partial\phi = 0$, and $\partial^2\sigma_{\phi\phi}/\partial\phi^2 < 0$, or

$$k_2(1 - 3\cos\phi_c) - k_1 \sin\phi_c = 0,$$
$$3k_2 \sin\phi_c - k_1 \cos\phi_c < 0. \qquad \text{(32a,b)}$$

If an energy balance type criterion is used to estimate the crack propagation load, one may need to calculate the strain energy release rate which is given in terms of k_1 and k_2 as follows [13]:

$$\frac{\partial U}{\partial a_0} = \frac{\pi(1+\kappa)}{4\mu}({k_1}^2 + {k_2}^2), \qquad (a_0 = (b-a)/2). \qquad (33)$$

4. THE RESULTS

The material combinations, the external loads, and the geometrical configurations used in the numerical examples are summarized in Table 1 (see Fig. 1 for the notation).

The materials roughly correspond to a metal-hard polymer combination (say, aluminum–epoxy) (cases A, B and D) and an elastic half plane (cases C, E and F). The crack is along

Table 1. The cases considered as numerical examples

	A		B		C		D		E		F	
	A_1	A_2	B_1	B_2	C_1	C_2	D_1	D_2	E_1	E_2	F_1	F_2
$p_1(r)$	$-p_0$	0	$-p_0$	0	$-p_0$	0	$-p_0$	0	$-p_0$	0	$-p_0$	0
$p_2(r)$	0	$-p_0$	0	$-p_0$	0	$-p_0$	0	$-p_0$	0	$-p_0$	0	$-p_0$
E_1/E_2	22·22		0·045		0		22·22		0		0	
ν_1	0·3		0·35				0·3					
ν_2	0·35		0·3				0·35					
Fixed	$d = 2a_0$		$d = 2a_0$		$d = 2a_0$		$c = 2a_0$		$c = 2a_0$		$\theta_0 = 40°$	
Variab.	θ_0		θ_0		θ_0		θ_0		θ_0		c	

the line ($\theta = \theta_0$, $a < r < b$) and the distances c, d and the crack length $2a_0$ are defined by (Fig. 1)

$$2a_0 = b - a, \qquad c = (a + b)/2, \qquad d = c \cos \theta_0 . \tag{34}$$

As indicated by Table 1, the results are obtained for uniform tractions

$$\tau_{2\theta\theta}(r, \theta_0) = p_1(r) = -p_0, \qquad \tau_{2r\theta}(r, \theta_0) = 0,$$

or

$$p_1(r) = 0, \qquad p_2(r) = -p_0, \qquad (a < r < b) \tag{35a,b}$$

applied to the crack surface. The results are given in Table 2. The quantities k_i', $(i = 1, 2)$ shown in the table are the stress intensity factor ratios defined by

$$k_i'(c_j) = k_i(c_j)/(p_0\sqrt{a_0}), \qquad (i, j = 1, 2; \quad c_1 = a, c_2 = b) \tag{36}$$

where $p_0\sqrt{a_0}$ is the stress intensity factor in a homogeneous infinite plane with a crack of length $2a_0$. The cleavage angle ϕ_c given in the table was obtained from (32). All results are obtained for the plane strain case† (i.e. $\kappa_j = 3 - 4\nu_j$, $j = 1, 2$).

In cases A, B and C the distance d from the interface is fixed as $d = 2a_0$ and the crack angle θ_0 is varied (see Fig. 1). The limiting cases $\theta_0 = 0$ and $\theta_0 = 90°$ of this problem were given in [7] and [4] and were reproduced with the present computer program for verification. Note that if the crack is in the less stiff material (i.e. $E_1 > E_2$, case A), generally there is a reduction, and if $E_1 < E_2$ (case B and C) there is an increase in the stress intensity factors compared to the values for the homogeneous medium.

In cases D and E the radial distance c of the crack center is fixed as $c = 2a_0$ and again θ_0 is varied. In case D where $E_1/E_2 = 22{\cdot}22$ for $\theta_0 = \pi/2$ crack becomes an interface crack for which the closed form solution is given by (e.g. [2, 3]:

$$f_2(t) + if_1(t) = \frac{\sigma_0 - i\tau_0}{\mu_1 b_2(1 + \gamma)}(t - 2i\beta)R(t), \qquad (|t| < 1),$$

$$(\tau_{2\theta\theta} - i\tau_{2r\theta})_{\theta_0 = \pi/2} = (\sigma_0 - i\tau_0)[(t - 2i\beta)R(t) - 1], \qquad (|t| > 1),$$

$$R(t) = \left(\frac{t + 1}{t - 1}\right)^{i\beta}(t^2 - 1)^{-1/2}, \qquad \beta = \frac{1}{2\pi}\log\left(\frac{1 + \gamma}{1 - \gamma}\right), \qquad \gamma = b_1/b_2,$$

$$b_1 = \frac{\mu_2}{\mu_2 + \kappa_2\mu_1} - \frac{\mu_2}{\mu_1 + \kappa_1\mu_2}, \qquad b_2 = \frac{\mu_2}{\mu_2 + \kappa_2\mu_1} + \frac{\mu_2}{\mu_1 + \kappa_1\mu_2},$$

$$t = (2r - b - a)/(b - a),$$

† See [7] for the comparison of the plane strain and the plane stress results.

$$
\begin{aligned}
k_1 - ik_2 &= \lim_{r\to b+0} (\tau_{2\theta\theta} - i\tau_{2r\theta})_{\theta_0=\pi/2} \left(\frac{r-b}{r-a}\right)^{i\beta} \sqrt{\frac{(r-a)(r-b)}{(b-a)/2}} \\
&= (\sigma_0 - i\tau_0)(1-2i\beta)\sqrt{(b-a)/2}, \\
&- (\sigma_0 + i\tau_0) = p_1(r) + ip_2(r),
\end{aligned} \tag{37}
$$

where it should be noted that for the uniform tractions considered in (37) the stress intensity factors at $r = a$ and $r = b$ are the same. It is seen that in this case the stress singularity is oscillating in character and the definition of the stress intensity factors k_1 and k_2 is slightly different. Hence, at $\theta_0 = \pi/2$ one would not expect the stress intensity factors and the cleavage angles to be continuous functions of θ_0. The quantity which is expected to be continuous in θ_0 is the strain energy release rate given by (33) for a homogeneous medium. For the interface crack this quantity is given by [14]

$$
\begin{aligned}
\left(\frac{\partial U}{\partial a_0}\right)_{12} &= \frac{\pi}{2} \frac{(\mu_1 + \kappa_1\mu_2)(\mu_2 + \kappa_2\mu_1)}{\mu_1\mu_2[(1+\kappa_1)\mu_2 + (1+\kappa_2)\mu_1]} (k_1^2 + k_2^2)_{12} \\
&= \frac{\pi}{2} \frac{1+\kappa_2}{\mu_2 a_{21}} (k_1^2 + k_2^2)_{12}.
\end{aligned} \tag{38}
$$

Thus, for the crack imbedded in medium 2 if we evaluate the strain energy release rate ratio as

$$
W_2 = \frac{4\mu_2}{\pi(1+\kappa_2)p_0^2 a_0} \left(\frac{\partial U}{\partial a_0}\right)_2 = k_1'^2 + k_2'^2, \tag{39}
$$

from (38) and

$$
\lim_{\theta_0\to\pi/2} \left(\frac{\partial U}{\partial a_0}\right)_2 = \left(\frac{\partial U}{\partial a_0}\right)_{12} \tag{40}
$$

we should have

$$
\lim_{\theta_0\to\pi/2} (k_1'^2 + k_2'^2) = \frac{2}{a_{21}} W_{21} = \frac{2}{a_{21}} \frac{k_1^2 + k_2^2}{p_0^2 a_0} = \frac{2}{a_{21}}(1+\beta). \tag{41}
$$

In case D, $a_{21} = 3{\cdot}93086$, $\beta = 0{\cdot}13420$ and the limit becomes $W_2 \to 0{\cdot}51796$ which is given in Table 2. If one plots W_2 vs θ_0 it may be seen that there is in fact a smooth transition from the imbedded crack to the interface crack.

In case E as $\theta_0 \to \pi/2$ the crack approaches the traction-free surface; hence, as shown in the Table, the stress intensity factors tend to infinity. It should be pointed out that the analysis given in this paper is for a crack imbedded into a homogeneous medium. In the half plane problem for $\theta_0 = \pi/2$ the crack disappears, there is a discontinuity in the solution, and hence the values given in the table for $\theta_0 = \pi/2$ simply indicate the trend.

The results given for case F show the effect of the radial distance c (Fig. 1) on the stress intensity factors k_i and the cleavage angle ϕ_c for a fixed value of the crack angle, $\theta_0 = 40°$ It is seen that as c increases the results approach the values corresponding to the infinite plane.

Table 2. The stress intensity factors and the probable crack propagation angles for the crack orientations, the material combinations, and the loads shown in Table 1

$(k_i' = k_i/p_0\sqrt{a_0},\ i=1, 2,\ a_0 = (b-a)/2)$

Case	θ_0	0	20°	40°	60°	80°	90°
A_1	$k_1'(a)$	0·9349	0·9307	0·9216	0·9144	0·9143	0·9160
	$k_1'(b)$	0·9617	0·9572	0·9457	0·9318	0·9206	0·9160
	$k_2'(a)$	0	0·0025	0·00001	−0·0071	−0·0153	−0·0188
	$k_2'(b)$	0	0·0125	0·0209	0·0237	0·0215	0·0188
	$\phi_c(a)$	0	−0·306	−0·001	0·889	1·911	2·342
	$\phi_c(b)$	0	−1·492	−2·531	−2·915	−2·666	−2·342
A_2	$k_1'(a)$	0	0·0223	0·0338	0·0333	0·0251	0·0191
	$k_1'(b)$	0	0·0030	0·0020	−0·0042	−0·0142	−0·0191
	$k_2'(a)$	0·9349	0·9395	0·9492	0·9580	0·9644	0·9663
	$k_2'(b)$	0·9617	0·9629	0·9655	0·9671	0·9672	0·9663
	$\phi_c(a)$	−70·53	−70·08	−69·85	−69·87	−70·03	−70·15
	$\phi_c(b)$	−70·53	−70·47	−70·49	−70·61	−70·81	−70·91
B_1	$k_1'(a)$	1·0780	1·0929	1·1165	1·1389	1·1459	1·1420
	$k_1'(b)$	1·0464	1·0571	1·0796	1·1091	1·1344	1·1420
	$k_2'(a)$	0	−0·0098	−0·0077	−0·0062	0·0242	0·0321
	$k_2'(b)$	0	−0·0220	−0·0377	−0·0433	−0·0382	−0·0321
	$\phi_c(a)$	0	1·024	0·793	−0·623	−2·417	−3·210
	$\phi_c(b)$	0	2·384	3·985	4·462	3·852	3·210
B_2	$k_1'(a)$	0	−0·0386	−0·0586	−0·0566	−0·0399	−0·0291
	$k_1'(b)$	0	−0·0082	−0·0092	−0·0002	0·0184	0·0291
	$k_2'(a)$	1·0780	1·0749	1·0590	1·0468	1·0400	1·0390
	$k_2'(b)$	1·0464	1·0465	1·0418	1·0385	1·0376	1·0390
	$\phi_c(a)$	−70·53	−71·22	−71·59	−71·56	−71·26	−71·06
	$\phi_c(b)$	−70·53	−70·68	−70·70	−70·53	−70·19	−70·00
C_1	$k_1'(a)$	1·0913	1·1049	1·1349	1·1604	1·1677	1·1621
	$k_1'(b)$	1·0539	1·0644	1·0918	1·1257	1·1543	1·1621
	$k_2'(a)$	0	−0·0114	−0·0093	0·0067	0·0276	0·0367
	$k_2'(b)$	0	−0·0251	−0·0431	−0·0498	−0·0439	−0·0367
	$\phi_c(a)$	0	1·184	0·941	−0·663	−2·703	−3·61
	$\phi_c(b)$	0	2·700	4·512	5·041	4·339	3·61
C_2	$k_1'(a)$	0	−0·0440	−0·0669	−0·0646	−0·0453	−0·0331
	$k_1'(b)$	0	−0·0095	−0·0109	−0·0003	0·0205	0·0331
	$k_2'(a)$	1·0913	1.0842	1·0686	1·0540	1·0452	1·0440
	$k_2'(b)$	1·0539	1·0523	1·0484	1·0444	1·0426	1·0440
	$\phi_c(a)$	−70·53	−71.31	−71·73	−71·70	−71·36	−71·05
	$\phi_c(b)$	−70·53	−70·70	−70·73	−70·54	−70·15	−69·93
D_1	$k_1'(a)$	0·9349	0·9201	0·8749	0·8034	0·7334	
	$k_1'(b)$	0·9617	0·9523	0·9217	0·8613	0·7753	
	$k_2'(a)$	0	0·0018	−0·0093	−0·0539	−0·1297	
	$k_2'(b)$	0	0·0139	0·0319	0·0586	0·0960	
	$\phi_c(a)$	0	−0·227	1·221	7·608	18·96	
	$\phi_c(b)$	0	−1·670	−3·956	−7·701	−13·72	
	$W(a)$	0·8740	0·8465	0·7655	0·6484	0·5546	0·5180
	$W(b)$	0·9248	0·9070	0·8505	0·7483	0·6104	0·5180

Table 2—*Continued*

Case	θ_0	0	20°	40°	60°	80°	90°
D_2	$k_1'(a)$	0	0·0259	0·0608	0·1124	0·1547	
	$k_1'(b)$	0	0·0029	−0·0008	−0·0258	−0·0919	
	$k_2'(a)$	0·9349	0·9299	0·9160	0·8910	0·8012	
	$k_2'(b)$	0·9617	0·9586	0·9493	0·9305	0·8598	
	$\phi_c(a)$	−70·53	−70·00	−69·27	−68.14	−66·89	
	$\phi_c(b)$	−70·53	−70·47	−70·55	−71·06	−72·58	
	$W(a)$	0·8740	0·8654	0·8427	0·8064	0·6658	0·5180
	$W(b)$	0·9248	0·9188	0·9012	0·8664	0.7476	0·5180
E_1	$k_1'(a)$	1·0913	1·1222	1·2437	1·6218	4·0850	$\to\infty$
	$k_1'(b)$	1·0539	1·0725	1·1468	1·3871	3·0851	$\to\infty$
	$k_2'(a)$	0	−0·0121	−0·0041	0·1125	1·4986	$\to\infty$
	$k_2'(b)$	0	−0·0287	−0·0762	−0·2099	−1·2715	$\to-\infty$
	$\phi_c(a)$	0	1·235	0·373	−7·859	−33·458	
	$\phi_c(b)$	0	3·063	7·540	16·494	36·013	
E_2	$k_1'(a)$	0	−0·0524	−0·1357	−0·3262	−0·9288	$\to-\infty$
	$k_1'(b)$	0	−0·0104	−0·0151	0·0056	0·1336	$\to\infty$
	$k_2'(a)$	1·0913	1·0979	1·1177	1·1381	1·2538	$\to\infty$
	$k_2'(b)$	1·0539	1·0589	1·0767	1·1187	1·2702	$\to\infty$
	$\phi_c(a)$	−70·53	−71·44	−72·86	−76·08	−84·91	
	$\phi_c(b)$	−70·53	−70·72	−70·80	−70·43	−68·53	
	c/a_0	1·1	1·2	1·4	1·6	2·0	4·0
F_1	$k_1'(a)$	2·5263	1·9959	1·5947	1·4150	1·2437	1·0543
	$k_1'(b)$	1·4694	1·3772	1·2764	1·2168	1·1468	1·0423
	$k_2'(a)$	0·2103	0·1091	0·0400	0·0141	−0·0041	−0·0073
	$k_2'(b)$	−0·2911	−0·2308	−0·1625	−0·1221	−0·0762	−0·0171
	$\phi_c(a)$	−9·388	−6·219	−2·870	−1·144	0·373	0·796
	$\phi_c(b)$	20·92	18·08	14·08	11·24	7·540	1·882
F_2	$k_1'(a)$	−1·0299	−0·6667	−0·3819	−0·2538	−0·1357	−0·0227
	$k_1'(b)$	−0·0716	−0·0491	−0·0300	−0·0219	−0·0151	−0·0066
	$k_2'(a)$	1·6185	1·4100	1·2580	1·1884	1·1177	1·0288
	$k_2'(b)$	1·2639	1·2054	1·1461	1·1134	1·0767	1·0227
	$\phi_c(a)$	−82·94	−79·73	−76·41	−74·65	−72·86	−70·95
	$\phi_c(b)$	−71·61	−71·31	−71·03	−70·91	−70·80	−70·65

The results given in Table 2 may be used to obtain the stress intensity factors in an arbitrarily loaded two-phase composite medium with an arbitrarily oriented crack provided the medium is loaded sufficiently far away from the crack region so that in the perturbed problem the crack surface tractions $p_1(r)$ and $p_2(r)$ can be approximated by uniform stresses. Thus if $p_1(r) + ip_2(r) = p_1 + ip_2 =$ constant, we find

$$k_i(c_m) = -\sqrt{a_0}\sum_{1}^{2} k'_{ij}(c_m)p_j, \qquad (i, j, m = 1, 2;\quad c_1 = a,\ c_2 = b) \tag{42}$$

where in k'_{ij} $i = 1$ and $i = 2$ respectively correspond to the normal and the shear components of the stress intensity ratio given in Table 2 as k_1' and k_2', and $j = 1$ and $j = 2$ respectively refer to the external loads $p_1 \neq 0, p_2 = 0$ and $p_1 = 0, p_2 \neq 0$. For example, if the medium is loaded parallel to the interface away from the crack region, the uniform stresses in the uncracked material will be related by

$$\tau_{1r\theta}(r, 0) = \tau_{2r\theta}(r, 0) = 0,$$
$$\frac{1 - \nu_1{}^2}{E_1} \tau_{1\theta\theta}(r, 0) = \frac{1 - \nu_2{}^2}{E_2} \tau_{2\theta\theta}(r, 0),$$
$$\tau_{1\theta\theta}(r, 0) = \sigma_1, \qquad \tau_{2\theta\theta}(r, 0) = \sigma_2, \tag{43}$$

where σ_1 and σ_2 are constant. In this case the stress intensity factors may be expressed as

$$\frac{k_1(c_m)}{\sigma_2\sqrt{a_0}} = k_1'(c_m) = k_{11}'(c_m)\cos^2\theta_0 + k_{12}'(c_m)\sin\theta_0\cos\theta_0,$$
$$\frac{k_2(c_m)}{\sigma_2\sqrt{a_0}} = k_2'(c_m) = k_{21}'(c_m)\cos^2\theta_0 + k_{22}'(c_m)\sin\theta_0\cos\theta_0, \qquad (m = 1, 2; \quad c_1 = a, c_2 = b). \tag{44}$$

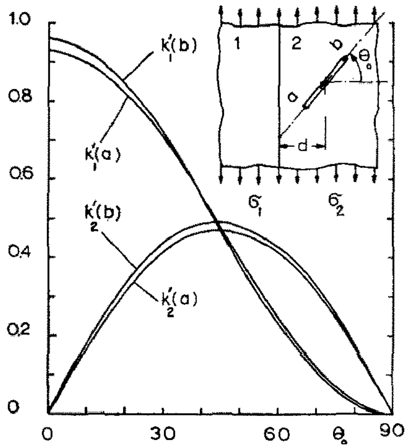

Fig. 2. Stress intensity factor ratios in bonded half planes containing an arbitrarily oriented crack and loaded parallel to the interface ($E_1/E_2 = 22{\cdot}22$, $\nu_1 = 0{\cdot}3$, $\nu_2 = 0{\cdot}35$, $d = 2a_0 = b - a =$ constant).

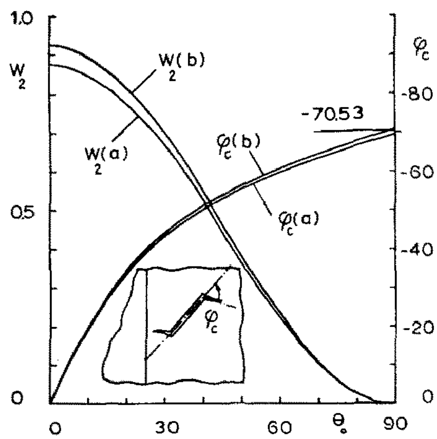

Fig. 3. Cleavage angles and the strain energy release rates for the bonded planes shown in Fig. 2 ($E_1/E_2 = 22{\cdot}22$, $\nu_1 = 0{\cdot}3$, $\nu_2 = 0{\cdot}35$, $d = 2a_0 = b - a =$ constant).

Figures 2–9 show some of the results obtained from (44). Figures 2 and 3 give the results for the material combination and the crack orientation corresponding to case A in Table 1. Similarly, (aside from the external loads which are given by (43)), Figs. 4 and 5 correspond to the case B, Figs. 6 and 7 correspond to the case C, and Figs. 8 and 9 correspond to the case D. Note that the probable cleavage angles ϕ_c shown in Figs. 3, 5, 7 and 9 are all negative and the direction of crack initiation is approximately perpendicular to the direction of the external load. The angle $\phi_c = 70°{\cdot}53$ shown in the figures corresponds to the cleavage angle for an infinite plane containing a crack and subjected to a uniform shear at infinity parallel and perpendicular to the plane of the crack [13]. For a fixed external load, the figures clearly show the effect of the crack orientation on the stress intensity factors, and hence, on the fracture resistance of the composite medium.

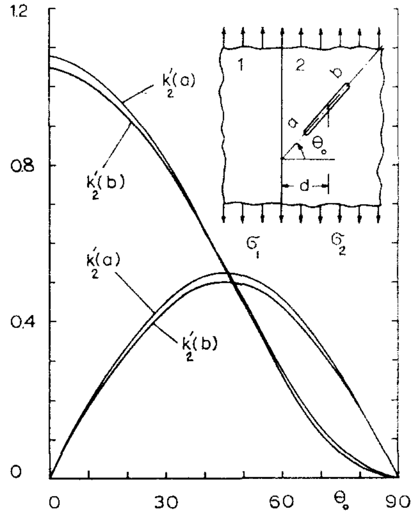

Fig. 4. Stress intensity factor ratios in bonded half planes containing an arbitrarily oriented crack and loaded parallel to the interface ($E_1/E_2 = 0{\cdot}045$, $\nu_1 = 0{\cdot}35$, $\nu_2 = 0{\cdot}3$, $d = 2a_0 = b - a =$ constant).

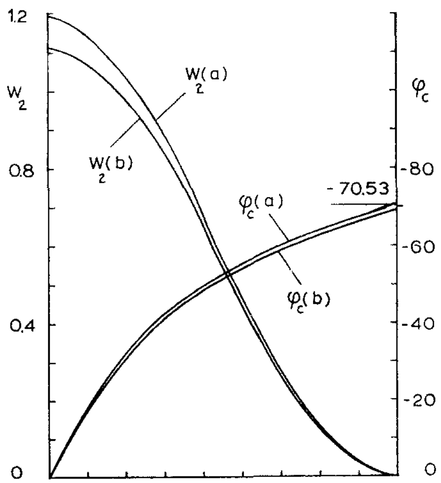

Fig. 5. Cleavage angles ϕ_c and the strain energy release rates W_2 for the bonded planes shown in Fig. 4 ($E_1/E_2 = 0{\cdot}045$, $\nu_1 = 0{\cdot}35$, $\nu_2 = 0{\cdot}3$, $d = 2a_0 = b - a =$ constant).

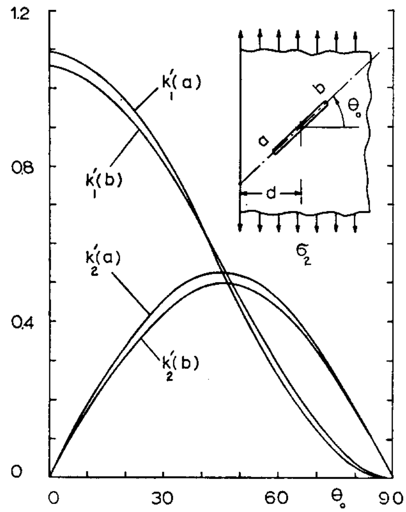

Fig. 6. Stress intensity factor ratios in a half plane containing an arbitrarily oriented internal crack and loaded parallel to the free boundary ($d = 2a_0 = b - a =$ constant).

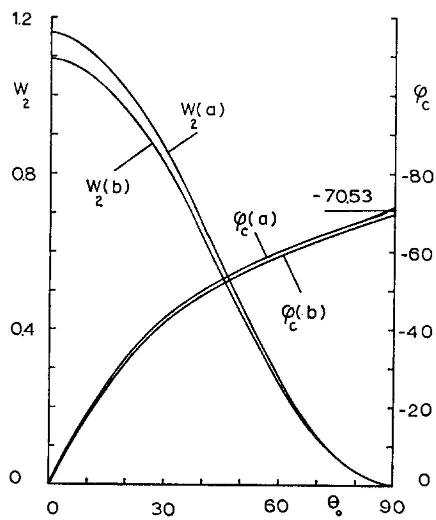

Fig. 7. ϕ_c and W_2 for the half plane shown in Fig. 6 ($d = 2a_0 = b - a =$ constant).

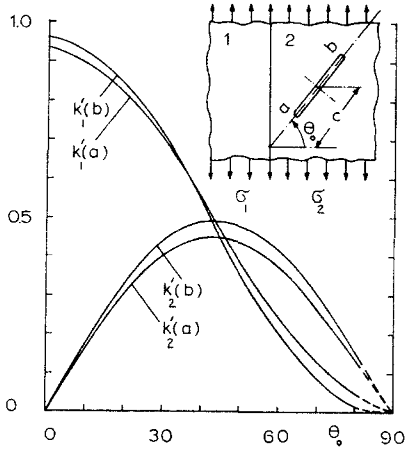

Fig. 8. Stress intensity factor ratios in bonded half planes containing an arbitrarily oriented crack and loaded parallel to the interface ($E_1/E_2 = 22{\cdot}22$, $\nu_1 = 0{\cdot}3$, $\nu_2 = 0{\cdot}35$, $c = 2a_0 = b - a =$ constant).

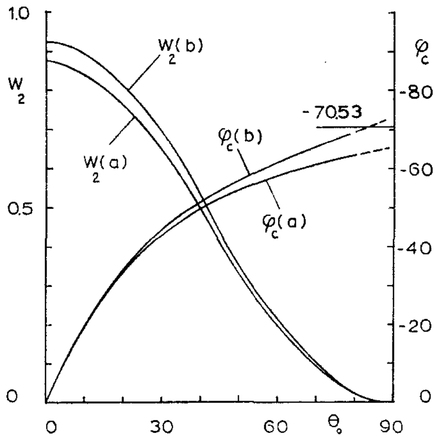

Fig. 9. ϕ_c and W_2 for the bonded planes shown in Fig. 8 ($E_1/E_2 = 22{\cdot}22$, $\nu_1 = 0{\cdot}3$, $\nu_2 = 0{\cdot}35$, $c = 2a_0 = b - a =$ constant).

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Абстракт—Рассматривается проблема эластостатической плоскости двух связанных полуплоскостей с произвольно расположенной трещиной вблизи границы раздела. Преобразованием Меллина проблема формулируется в виде сингулярных интегральных уравнений. При помощи уравнений разрешаются вопросы ориентации различных трещин, комбинации материалов и наружных нагрузок. Численные результаты приведенные в этой работе включают коэффициент силы напряжения, коэффициент выделения потенциальной энергии деформации и возможные углы расщепления, дающие направление развитию трещин.